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Projected Tikhonov regularization method for Fredholm integral equations of the first kind

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Abstract
In this paper, we consider a variant of projected Tikhonov regularization method for solving Fredholm integral equations of the first kind. We give a theoretical analysis of this method in the Hilbert space $L^2(a,b)$ setting and establish some convergence rates under certain regularity assumption on the exact solution and the kernel $k(\cdot,\cdot)$. Some numerical results are also presented.

Keywords: ill-posed problems; integral equation of the first kind; projected Tikhonov regularization method

1 Introduction

Let $H = L^2((a,b);\mathbb{R})$ and consider the Fredholm integral equation of the first kind

$$\int_a^b k(t,s)f(s)\,ds = g(t), \qquad t \in [a,b], \tag{1}$$

where $k(\cdot,\cdot)$ and g are known functions, and f is the unknown function to be determined. The equation can be written as an operator equation

$$K : H \longrightarrow H, \qquad f \longmapsto g = Kf. \tag{2}$$

Many inverse problems in applied science and engineering (see, e.g., [1–3] and references therein) lead to the solution of Fredholm integral equations of the first kind (1).

Several numerical methods are available in the literature to solve linear integral equations of the first kind; we can cite, for example, multiscale methods [4–8], spectral-collocation methods [9, 10], reproducing kernel Hilbert space methods [11, 12], eigenvalue approximation methods [13–17], quadrature-based collocation methods [18, 19], projections methods [20–28], and other interesting methods quite exposed in the books [13, 29–36].

In the regularizing procedures, several authors have studied finite-dimensional approximations obtained by projecting regularized approximations into finite-dimensional subspaces. Such methods may be called regularization projection methods.

The main idea of regularization by projection is to project the least squares minimization on a finite-dimensional subspace to obtain a well-conditioned problem and, consequently, a stabilization of the generalized inverse of the approximate operator. We can distinguish two different cases of regularization by projection. The first one is the regularization in preimage space, and the second is regularization in image space; see, for example, [21, 22, 26, 27, 33, 37].

Following the idea developed in [18, 19] and [26, 27], we analyze a variant of projected Tikhonov regularization method applied to our problem (2) in the Hilbert space $L^2(a,b)$ setting. We develop the theoretical framework of this method of approximation and give some results of convergence under certain conditions of regularity on the kernel $k(\cdot,\cdot)$ and the solution of the problem in question.

More precisely, we build a method of projection by using very simple mathematical tools, which can be concretized and implemented numerically. Moreover, we give natural conditions on the kernel $k(\cdot,\cdot)$ of the operator K, which enables us to establish the convergence results of this approach. For the subspace of projection, we use the Legendre polynomials, which are well studied in the literature compared to other classes of polynomials. This judicious choice also enables us to give a simple calculation and explicit formula of approximation of K^*K (see (25)). It is important to note that in [27], the author gives sufficient conditions on $\|A - A_n\|$ within an abstract framework to establish the convergence of this approximation, which returns an approach very limited in practice; moreover, it is not exploitable from the numerical point of view.

In this investigation, we assume that

(A1) $k(\cdot,\cdot)$ is nondegenerate.

(A2) $k(\cdot,\cdot) \in L^2((a,b)\times(a,b);\mathbb{R})$, that is, $\kappa^2 = \int_a^b \int_a^b |k(t,s)|^2\,dt\,ds < +\infty$.

It is well known that under these conditions, K is a compact (Hilbert-Schmidt) integral operator with infinite-dimensional range ($\dim(\mathcal{R}(K)) = +\infty$). In this case, $\mathcal{R}(K)$ is not closed, and problem (2) belongs to the class of ill-posed problems. The ill-posedness character means that $T^\dagger$ (the Moore-Penrose inverse) or K^{-1} (when K is injective) are unbounded operators. Consequently, the standard numerical procedures to solve such equations are unstable and pose very serious problems when the data are not exact; that is, small perturbations of the observation data may lead to large changes on the considered solution.

To overcome this difficulty and for obtaining stable approximate solutions for ill-posed problems, regularization procedures are employed, and Tikhonov regularization is one the such procedure. This method consists in minimizing over H the so-called Tikhonov functional

$$\Phi_\alpha(f) = \|Kf - g\|_H^2 + \alpha\|f\|_H^2,$$

where $\alpha > 0$ is the regularization parameter. The regularized solution f_α is the unique minimizer of the Tikhonov functional $\Phi_\alpha(f)$. We denote this minimum by $f_\alpha = \arg\min_{f\in H} \Phi_\alpha(f)$, which is a unique solution of the normal equation

$$\bigl(\alpha I + K^*K\bigr)f = K^*g.$$

The linear operator $R(\alpha) = (\alpha I + K^*K)^{-1}K^* \in \mathcal{L}(H)$ is called a regularizing operator, and we have

$$\left\| \left(\alpha I + K^*K\right)^{-1} K^* \right\|_H^2 = \frac{1}{2\sqrt{\alpha}}, \tag{3}$$

$$\|f - f_\alpha\|_H^2 \longrightarrow 0, \quad \alpha \longrightarrow 0. \tag{4}$$

To establish the main results of our work, we introduce the following assumptions:

(H1) The operator K is injective, that is, $N(K) = \{0\}$.
(H2) $g \in \mathcal{R}(K)$.
(H3) The kernel $k(\cdot,\cdot) \in \mathcal{C}^r([a,b] \times [a,b];\mathbb{R})$, $r \in \mathbb{N}$.
(H4) The operator K^* is injective ($\Longleftrightarrow \overline{\mathcal{R}(K)} = H$).

2 Preliminaries and notation

In this section, we present the notation and functional setting and prepare some material, which will be used in our analysis. For more details, we refer the reader to [32, 36, 38].

Let H_1 and H_2 be two real Hilbert spaces. We denote by $\mathcal{L}(H_1, H_2)$ the space of all bounded linear operators from H_1 to H_2 (and $\mathcal{L}(H)$ if $H_1 = H_2 = H$) with the operator norm

$$\|T\| = \sup_{\|u\|_{H_1} \le 1} \|Tu\|_{H_2}, \quad T \in \mathcal{L}(H).$$

The null-space of $T \in \mathcal{L}(H_1, H_2)$ is the set $\mathcal{N}(T) = \{u \in H_1 : Tu = 0\}$, whereas the range of T is denoted by $\mathcal{R}(T) = T(H_1) = \{v = Tu, u \in H_1\}$.

Let $T \in \mathcal{L}(H_1, H_2)$. Recall that, for $v \in H_2$, the linear operator equation

$$Tu = v \tag{5}$$

has a solution if and only if $v \in \mathcal{R}(T)$.

• If $\mathcal{R}(T)$ is infinite-dimensional and T is injective, then $T^{-1} : \mathcal{R}(T) \longrightarrow H_1$ is bounded if and only if $R(T)$ is closed.

• If $v \notin \mathcal{R}(T)$, then we look for an element $\hat{u} \in H_1$ such that $T\hat{u}$ is "closest to" v in the sense that $\hat{u}$ minimizes the functional $\|Tu - v\|_{H_2}$.

Definition 2.1 Let $T \in \mathcal{L}(H_1, H_2)$. We call $\hat{u} \in H_1$ a least residual norm solution (LRN solution) of (5) if

$$\|T\hat{u} - v\|_{H_2} = \inf_{u \in H_1} \|Tu - v\|_{H_2}.$$

Definition 2.2 For $v \in G = \mathcal{R}(T) + \mathcal{R}(T)^\perp$, we denote the set of all LRN solutions of (5) by

$$S_v = \left\{ \hat{u} \in H_1 : \|T\hat{u} - v\|_{H_2} = \inf_{u \in H_1} \|Tu - v\|_{H_2} \right\}.$$

Definition 2.3 Let $v \in G = \mathcal{R}(T) + \mathcal{R}(T)^\perp$. Then $u^\dagger \in S_v$ is called a best approximate solution (generalized solution) of (5) if $\|u^\dagger\|_{H_1} = \inf_{\hat{u} \in S_v} \|\hat{u}\|_{H_1}$.

Theorem 2.1 *Let $v \in G = \mathcal{R}(T) + \mathcal{R}(T)^{\perp}$. Then there exist a unique $x^{\dagger} \in S_v$ such that*

$$\|u^{\dagger}\|_{H_1} = \inf_{\hat{u} \in S_v} \|\hat{u}\|_{H_1},$$

and

$$u^{\dagger} \in \mathcal{N}(A)^{\perp}, \quad u^{\dagger} = P\hat{u}_0,$$

where $P : H_1 \longrightarrow \mathcal{N}(A)^{\perp}$ is the orthogonal projection onto $\mathcal{N}(A)^{\perp}$ and $\hat{u}_0$ is any element in S_v.

Definition 2.4 The Moore-Penrose (generalized) inverse $T^{\dagger} : D(T^{\dagger}) \longrightarrow H_1$ of T defined on the dense domain $D(T^{\dagger}) = \mathcal{R}(T) + \mathcal{R}(T)^{\perp}$ maps $v \in D(T^{\dagger})$ to the best-approximate solution of (5), that is, $T^{\dagger} v = u^{\dagger}$.

Remark 2.1
- $T^{\dagger} = T^{-1}$ if $\mathcal{R}(T)^{\perp} = \{0\}$ and $\mathcal{N}(T) = \{0\}$.
- $T^{\dagger}$ is continuous if and only if $\mathcal{R}(T)$ is a closed subspace of H_2.

Theorem 2.2 ([18], Thm. 2.7, p.31) *Let E, F be two Banach spaces, and $(T_n) \subset \mathcal{L}(E,F)$. Then, $T_n \longrightarrow T \in \mathcal{L}(E,F)$ pointwise (i.e., $T_n x \longrightarrow Tx$ for all $x \in E$) if and only if the sequence (T_n) is uniformly bounded, and $T_n x \longrightarrow Tx$ for all $x \in \mathcal{D}$, where $\mathcal{D} \subset E$ is a dense subspace of E.*

We denote by $(\lambda_i, e_i)_{i=0}^{\infty}$ the normalized eigensystem of the compact self-adjoint operator $A = K^*K$. Then A can be diagonalized according to the following formula:

$$h = \sum_{i=1}^{\infty} \langle h, e_i \rangle e_i, \quad Ah = \sum_{i=1}^{\infty} \lambda_i \langle h, e_i \rangle e_i. \tag{6}$$

The classical Legendre polynomials $(L_j)_{j\in\mathbb{N}}$ are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae:

$$\begin{cases} L_0(x) = 1, \qquad L_1(x) = x, \\ L_{j+1}(x) = (\frac{2j+1}{j+1}) x L_j(x) - (\frac{j}{j+1}) L_{j-1}(x), \quad j = 1, 2, \ldots. \end{cases} \tag{7}$$

In order to use these polynomials on the interval $[a,b]$, we define the so-called normalized shifted Legendre polynomials of degree n as follows: Let $x \in [a,b]$; then the transformation $y = \frac{2}{b-a}x - \frac{a+b}{b-a}$ transforms the interval $[a,b]$ onto $[-1,1]$ and the normalized shifted Legendre polynomials are given by

$$\hat{L}_j(x) = \sqrt{\frac{2}{b-a}} \sqrt{\frac{2j+1}{2}} L_j\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right), \quad x \in [a,b], j \in \mathbb{N}. \tag{8}$$

The set $(\hat{L})_{j\in\mathbb{N}}$ is a complete orthonormal system in $H = L^2((a,b);\mathbb{R})$, namely

$$\langle \hat{L}_j, \hat{L}_i \rangle = \int_a^b \hat{L}_j(x) \hat{L}_j(x)\, dx = \delta_{ji}, \tag{9}$$

where δ_{ji} is the Kronecker symbol.

Thus, for any function $h \in H = L^2((a,b);\mathbb{R})$, we have the Fourier-Legendre expansion

$$h = \sum_{j=0}^{\infty} c_j(h)\hat{L}_j, \tag{10}$$

where the Fourier-Legendre coefficients $c_j(h)$ are given by

$$c_j(h) = \langle h, \hat{L}_j \rangle = \int_a^b \hat{L}_j(x)h(x)\,dx, \quad j \in \mathbb{N}.$$

3 Projected Tikhonov regularization method

Let $H_n = \operatorname{span}\{\hat{L}_j, j = 0,1,\dots,n\}$ be the sequence of Legendre polynomial subspaces of degree $\leq n$, and let $\Pi_n : H \longrightarrow H_n$ be the orthogonal projection defined as

$$\Pi_n h = \sum_{j=0}^{n} c_j(h)\hat{L}_j, \quad h \in H. \tag{11}$$

We quote some crucial properties of Π_n ([38], pp.283-287 and [39]).

Lemma 3.1 *Let Π_n be the orthogonal projection defined in* (11). *Then we have*

$$\forall u \in H, \quad \|(I - \Pi_n)h\|_{L^2(a,b)} \longrightarrow 0, \quad n \longrightarrow \infty, \tag{12}$$

$$\forall u \in C^r([a,b];\mathbb{R}), \quad \|(I - \Pi_n)u\|_{L^2(a,b)} \leq cn^{-r}\|u^{(r)}\|_{L^2(a,b)}, \tag{13}$$

$$\forall u \in C^r([a,b];\mathbb{R}), \quad \|(I - \Pi_n)u\|_{\infty} \leq cn^{\frac{3}{4}-r}\|u^{(r)}\|_{L^2(a,b)}, \tag{14}$$

where c is a positive constant independent of n, and r is a positive integer.

Remark 3.1 Let $\mathbb{N} = \{0,1,2,\dots\}$ and $r \in \mathbb{N}$.

1. If $k(\cdot,\cdot) \in \mathcal{C}^r([a,b] \times [a,b];\mathbb{R})$, then $\mathcal{R}(K) \subset \mathcal{C}^r([a,b] \times [a,b];\mathbb{R})$. Further, denoting $D_{i,j}k(t,s) = \frac{\partial^{i+j}}{\partial t^i\,\partial s^j}k(t,s)$, we have

$$\begin{aligned}\|D_{i,j}k\|_{r,\infty} &= \sup_{0\leq i+j\leq r} \|D_{i,j}k\|_{\infty} \\ &= \sup_{0\leq i+j\leq r} \Big\{ \sup_{(t,s)\in[a,b]\times[a,b]} \big|D_{i,j}k(t,s)\big| \Big\} \\ &= \sup_{0\leq i+j\leq r} M_{i,j} = M_r < \infty. \end{aligned}\tag{15}$$

2. If $f \in L^2((a,b);\mathbb{R})$ and $k(\cdot,\cdot) \in \mathcal{C}^r([a,b] \times [a,b];\mathbb{R})$, then we have the following estimates

$$\begin{aligned}\big|D_i(Kf)(t)\big| &= \left|\frac{d^i}{dt^i}(Kf)(t)\right| = \left|\int_a^b \frac{\partial^i}{\partial t^i}k(t,s)f(s)\,ds\right| \\ &\leq M_{i,0}\int_a^b \big|f(s)\big|\,ds \leq M_{i,0}\sqrt{(b-a)}\|f\|_{L^2(a,b)}, \end{aligned}\tag{16}$$

which leads to

$$\big\|D_i(Kf)\big\|_{L^2((a,b))} \le M_{i,0}(b-a)\|f\|_{L^2(a,b)}, \quad i = 0,1,\dots,r, \tag{17}$$

$$\big\|D_i(Kf)\big\|_\infty \le M_{i,0}\sqrt{(b-a)}\|f\|_{L^2(a,b)}, \quad i = 0,1,\dots,r. \tag{18}$$

In practice, ill-posed problems like integral equations of the first kind have to be approximated by a finite-dimensional problem whose solution can be easy calculated by using some numerical computation software.

In this paper, we replace the original problem $Kf = g$ by an algebraic system $K_n f^n = g_n$ posed on $\mathbb{R}^{n+1}$, where the Moore-Penrose generalized inverse $K_n^\dagger$ is defined for every data $g_n \in \mathbb{R}^{n+1}$.

We define the linear operator $\mathbb{Q}_n : H = L^2(a,b) \longrightarrow \mathbb{R}^{n+1}$:

$$\forall f = \sum_{j=0}^{\infty} c_j(f)\hat{L}_j \in H, \quad \mathbb{Q}_n f = \big(c_0(f), c_1(f), \dots, c_n(f)\big)^T. \tag{19}$$

Now, the original equation (2) is replaced by an operator equation in $\mathbb{R}^{n+1}$, which can be written abstractly as

$$K_n : H \longrightarrow \mathbb{R}^{n+1}, \qquad K_n f = (\mathbb{Q}_n K)f = \mathbb{Q}_n g = g_n. \tag{20}$$

Theorem 3.1 *Let $K_n : H = L^2((a,b);\mathbb{R}) \longrightarrow \mathbb{R}^{n+1}$ be given by formula* (20). *Then, K_n is a bounded operator, and the adjoint $K_n^* : \mathbb{R}^{n+1} \longrightarrow H = L^2((a,b);\mathbb{R})$ of K_n is given by*

$$\begin{cases} \forall f \in H, \forall X = (x_0, x_1, \dots, x_n)^T \in \mathbb{R}^{n+1}, \quad \langle K_n f, X\rangle_{\mathbb{R}^{n+1}} = \langle f, K_n^* X\rangle_H, \\ (K_n^* X)(t) = \sum_{j=0}^{n} x_j (K^*\hat{L}_j)(t), \end{cases} \tag{21}$$

where K^* is the adjoint of K:

$$\big(K^* u\big)(t) = \int_a^b k^*(t,s)u(s)\,ds = \int_a^b k(s,t)u(s)\,ds.$$

Proof (1) For every $f \in H$, we have

$$\begin{aligned} \|K_n f\|^2_{\mathbb{R}^{n+1}} &= \sum_{j=0}^{n} \big|c_j(Kf)\big|^2 \\ &= \sum_{j=0}^{n} \big|\langle Kf, \hat{L}_j\rangle\big|^2_H \\ &\le \sum_{j=0}^{\infty} \big|\langle Kf, \hat{L}_j\rangle\big|^2_H = \|Kf\|^2_H \\ &\le \kappa^2 \|f\|_2^2, \end{aligned} \tag{22}$$

which implies that $K_n \in \mathcal{L}(H, \mathbb{R}^{n+1})$ and $\|K_n\| \le \kappa = (\int_a^b \int_a^b |k(t,s)|^2\,dt\,ds)^{\frac{1}{2}}$.

(2) By definition of $\mathbb{Q}_n : H = L^2(a,b) \longrightarrow \mathbb{R}^{n+1}$ it is easy to check that $\mathbb{Q}_n \in \mathcal{L}(H, \mathbb{R}^{n+1})$. Thus, we can define its adjoint operator $\mathbb{Q}_n^* : \mathbb{R}^{n+1} \longrightarrow H = L^2(a,b)$. Now, for

$$\mathbb{Q}_n f = \left(\langle f, \hat{L}_0\rangle_{L^2(a,b)}, \ldots, \langle f, \hat{L}_n\rangle_{L^2(a,b)}\right)^{\perp} \in \mathbb{R}^{n+1}, \quad X = (x_0, \ldots, x_n)^{\perp} \in \mathbb{R}^{n+1},$$

from the identity

$$\langle \mathbb{Q}_n f, X\rangle_{\mathbb{R}^{n+1}} = \sum_{j=0}^{n} x_j \langle f, \hat{L}_j\rangle_{L^2(a,b)} = \left\langle f, \sum_{j=0}^{n} x_j \hat{L}_j \right\rangle_{L^2(a,b)} = \left\langle f, \mathbb{Q}_n^* X\right\rangle_{L^2(a,b)}$$

it follows that

$$\mathbb{Q}_n^* X = \sum_{j=0}^{n} x_j \hat{L}_j, \tag{23}$$

$$K_n^* X = (\mathbb{Q}_n K)^* X = K^* \mathbb{Q}_n^* X = \sum_{j=0}^{n} x_j K^* \hat{L}_j, \tag{24}$$

and

$$\left(K_n^* K_n\right) f = \sum_{j=0}^{n} \langle Kf, \hat{L}_j\rangle_{L^2(a,b)} K^* \hat{L}_j. \tag{25}$$

□

Remark 3.2 The expression $(K_n^* X)(t) = \sum_{j=1}^{n} x_j (K^* \hat{L}_j)(t)$ allows us to conclude that

$$\mathcal{R}\left(K_n^*\right) = \operatorname{span}\left\{K^* \hat{L}_j, j = 0, 1, \ldots, n\right\}, \quad \dim\left(\mathcal{R}\left(K_n^*\right)\right) \le n + 1. \tag{26}$$

Since K_n^* is of finite rank, H can be written as

$$H = L^2(a,b) = \overline{\mathcal{R}\left(K_n^*\right)} \oplus \mathcal{N}(K_n) = \mathcal{R}\left(K_n^*\right) \oplus \mathcal{N}(K_n). \tag{27}$$

Here and in what follows, we denote $A = K^*K$ and $A_n = K_n^* K_n$. Note that K^* and K^*K are defined by

$$\left(K^* u\right)(t) = \int_a^b k^*(t,s) u(s)\, ds, \quad k^*(t,s) = \overline{k(s,t)} = k(s,t),$$

$$\left(K^* K u\right)(t) = \int_a^b \theta(t,s) u(s)\, ds, \quad \theta(t,s) = \int_a^b k(\tau, t) k(\tau, s)\, d\tau.$$

Now, we are in the position to prove our main results. In the following theorem, we show the convergence of A_n to A and also other regularizing properties of A_n.

Theorem 3.2 *Let $A = K^*K$ and $A_n = K_n^* K_n$ be given by expression* (25). *Then, under the assumption*

$$k(\cdot,\cdot) \in L^2\left((a,b) \times (a,b); \mathbb{R}\right), \tag{A2}$$

we have

$$\forall h \in H, \quad \|Ah - A_n h\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{28}$$

Moreover, if

$$k(\cdot,\cdot) \in \mathcal{C}^r\big([a,b] \times [a,b]; \mathbb{R}\big), \quad r \geq 1, \tag{H3}$$

then we have

$$\|A - A_n\| \leq \varepsilon(n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{29}$$

Proof We compute

$$\begin{aligned}
\big\|\big(K^*K - K_n^*K_n\big)h\big\|_{L^2(a,b)} &= \left\|K^*Kh - \sum_{j=0}^{n} c_j(Kh)\big(K^*\hat{L}_j\big)\right\|_{L^2(a,b)} \\
&= \left\|K^*\left(Kh - \sum_{j=0}^{n} c_j(Kh)\hat{L}_j\right)\right\|_{L^2(a,b)} \\
&= \big\|K^*(Kh - \Pi_n Kh)\big\|_{L^2(a,b)} \\
&\leq \big\|K^*\big\| \big\|(Kh - \Pi_n Kh)\big\|_{L^2(a,b)} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{30}$$

By Lemma 3.1((17)) we can write

$$\begin{aligned}
\big\|\big(K^*K - K_n^*K_n\big)h\big\|_{L^2(a,b)} &= \left\|K^*Kh - \sum_{j=0}^{n} c_j(Kh)\big(K^*\hat{L}_j\big)\right\|_{L^2(a,b)} \\
&= \left\|K^*\left(Kh - \sum_{j=0}^{n} c_j(Kh)\hat{L}_j\right)\right\|_{L^2(a,b)} \\
&= \big\|K^*(Kh - \Pi_n Kh)\big\|_{L^2(a,b)} \\
&\leq \big\|K^*\big\| \big\|(Kh - \Pi_n Kh)\big\|_{L^2(a,b)} \\
&\leq \|K\|\big(cn^{-r}\big)\big\|(Kh)^{(r)}\big\| \\
&\leq \|K\|\big(cn^{-r}\big)(b-a)M_{r,0}\|h\|_{L^2(a,b)} = \varepsilon(n)\|h\|_{L^2(a,b)},
\end{aligned} \tag{31}$$

which implies that

$$\|A - A_n\| \leq \|K\|\big(cn^{-r}\big)(b-a)M_{r,0} = \varepsilon(n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{32}$$

□

Lemma 3.2 *Let* $\alpha > 0$, $\mathbb{R}_n(\alpha) = (\alpha I + A_n)^{-1}A_n$, *and* $\mathbb{R}(\alpha) = (\alpha I + A)^{-1}A$. *Then*

$$\forall h \in H = L^2(a,b), \quad \big\|\mathbb{R}_n(\alpha)h - \mathbb{R}(\alpha)h\big\|_{L^2(a,b)} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{33}$$

Proof Before starting the proof, we recall the following useful result.

Remark 3.3 If K is a bounded injective operator, then

$$\mathcal{N}(K) = \mathcal{N}(K^*K) = \{0\} \quad \text{and} \quad \overline{R(K^*K)} = \mathcal{N}(K^*K)^{\perp} = \{0\}^{\perp} = H.$$

In view of Theorem 2.2 and Remark 3.3, to show the convergence result (33), it suffices to establish the result for $h \in \mathcal{R}(K^*K)$. Before starting the demonstration, we introduce the following propositions.

Proposition 3.1 *For all $h \in H$, we have*

$$\left\|(\alpha I + A)^{-1}Ah - h\right\|_H \longrightarrow 0, \quad \alpha \longrightarrow 0. \tag{34}$$

Proof If $h = \sum_{i=1}^{\infty} h_i e_i = \sum_{i=1}^{\infty} \langle h, e_i\rangle e_i$, then

$$\left\|(\alpha I + A)^{-1}Ah - h\right\|_H^2 = \sum_{i=1}^{\infty} \left(\frac{\alpha\lambda_i}{\alpha + \lambda_i}\right)^2 |h_i|^2.$$

For $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} |h_i|^2 \leq \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned}
\sum_{i=1}^{\infty} \left(\frac{\alpha}{\alpha + \lambda_i}\right)^2 |h_i|^2 &= \sum_{i=1}^{N} \left(\frac{\alpha}{\alpha + \lambda_i}\right)^2 |h_i|^2 + \sum_{i=N+1}^{\infty} \left(\frac{\alpha}{\alpha + \lambda_i}\right)^2 |h_i|^2 \\
&\leq \sum_{i=1}^{N} \left(\frac{\alpha}{\alpha + \lambda_i}\right)^2 |h_i|^2 + \frac{\varepsilon}{2} \\
&\leq \sum_{i=1}^{N} \left(\frac{\alpha}{\lambda_i}\right)^2 |h_i|^2 + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \alpha^2 \frac{1}{\lambda_N^2} \|h\|_H^2.
\end{aligned}$$

If we choose the parameter α such that $\alpha^2 \frac{1}{\lambda_N^2} \|h\|_H^2 \leq \frac{\varepsilon}{2}$, then we obtain the desired convergence. □

Proposition 3.2 *We have*

$$\forall n \in \mathbb{N}, \quad \left\|(\alpha I + A_n)^{-1}A_n\right\| = \sup_{\lambda \in [0, \|A_n\|]} \frac{\lambda}{\alpha + \lambda} \leq 1, \tag{35}$$

that is, the sequence $(\mathbb{R}_n(\alpha))$ is uniformly bounded with respect to n.

We return now to the proof of Lemma (3.2). We have

$$\begin{aligned}
\left\|\mathbb{R}_n(\alpha)h - \mathbb{R}(\alpha)h\right\|_{L^2(a,b)} &= \left\|(\alpha I + A_n)^{-1}\left[(\alpha I + A_n)A - A_n(\alpha I + A)\right](\alpha I + A)^{-1}h\right\|_{L^2(a,b)} \\
&= \left\|\alpha(\alpha I + A_n)^{-1}(A - A_n)(\alpha I + A)^{-1}h\right\|_{L^2(a,b)} \\
&\leq \alpha\left\|(\alpha I + A_n)^{-1}\right\| \left\|(A - A_n)(\alpha I + A)^{-1}h\right\|_{L^2(a,b)}.
\end{aligned}$$

Using the fact that $\alpha\|(\alpha I + A_n)^{-1}\| \leq 1$, we derive

$$\left\|\mathbb{R}_n(\alpha)h - \mathbb{R}(\alpha)h\right\|_{L^2(a,b)} \leq \left\|(A - A_n)(\alpha I + A)^{-1}h\right\|_{L^2(a,b)}, \tag{36}$$

and from (30) and (34) we deduce that this last inequality tends to 0 for all $h \in \mathcal{R}(A)$. □

4 Convergence and error analysis

We denote by $R(\alpha) = (\alpha I + K^*K)^{-1}K^* \in \mathcal{L}(H)$ (resp. $(\alpha I + K_n^*K_n)^{-1}K_n^*$) the regularizing operator of K (resp. of K_n).

To establish the convergence results of this method, we point out the following results:

$$\left\|\left(\alpha I + K_n^*K_n\right)^{-1}K_n^*\right\| = \left\|\left(\alpha I + K^*K\right)^{-1}K^*\right\| = \frac{1}{2\sqrt{\alpha}}; \tag{37}$$

if $f \in H$, then

$$\|f - f_\alpha\|_{L^2(a,b)} \longrightarrow 0, \quad \alpha \longrightarrow 0; \tag{38}$$

also, if $f = Au \in R(A)$, then

$$\|f - f_\alpha\|_{L^2(a,b)} \leq \alpha\|u\|_{L^2(a,b)}. \tag{39}$$

Let us assume that g_δ are observation data of g such that

$$\|g - g_\delta\|_{L^2(a,b)} = \left(\sum_{j=0}^{\infty}\left\|c_j(g - g_\delta)\right\|^2\right)^{\frac{1}{2}} \leq \delta \tag{40}$$

with a given noise level $\delta > 0$. Then we have

$$\left\|g_n - g_n^\delta\right\|_{\mathbb{R}^{n+1}} \leq \left\|g - g^\delta\right\|_{L(a,b)} \leq \delta. \tag{41}$$

Let us consider the following equations:

$$K_nf^n = g_n, \tag{42}$$

$$K_nf^{\delta,n} = g_n^\delta, \tag{43}$$

$$\left(\alpha I + K^*K\right)f = K^*g, \tag{44}$$

$$\left(\alpha I + K^*K\right)f = K^*g^\delta. \tag{45}$$

Because our original problem ($Kf = g$) is ill-posed, the problem of finding the generalized solution $f^{\dagger,\delta,n} = K_n^\dagger g_n^\delta \in \mathcal{N}(K_n)^T$ of problem (42) with inexact data g_n^δ is instable. Regularizing this equation by the Tikhonov regularization method, we obtain

$$\left(\alpha I + K_n^*K_n\right)f = K_n^*g_n, \tag{46}$$

$$\left(\alpha I + K_n^*K_n\right)f = K_n^*g_n^\delta. \tag{47}$$

Denote by $f = K^{-1}g$ the exact solution of (2), by f_α (resp. f_α^δ) the regularized solution of (44) (resp. of (45)), and by f_α^n (resp. $f_\alpha^{n,\delta}$) the regularized solution of (46) (resp. of (47)).

Definition 4.1 We denote by f_α (resp. f_α^δ) the regularized solution of problem (2) for the exact data g (resp. for the inexact data g_δ):

$$f_\alpha = R(\alpha)g = (\alpha I + K^*K)^{-1}K^*g, \tag{48}$$

$$f_\alpha^\delta = R(\alpha)g_\delta = (\alpha I + K^*K)^{-1}K^*g_\delta. \tag{49}$$

Definition 4.2 For any $\alpha > 0$, the unique solution $f_\alpha^{\delta,n}$ of (47) is considered as a regularized solution of $f^{\dagger,n,\delta}$.

Remark 4.1 Without loss of generality, we can assume that $\dim(\mathcal{R}(K_n^*)) = n+1$. For example, under condition (H4), the vectors $K^*\hat{L}_j$, $i = 0,1,\dots,n$, are linearly independent, and consequently $\dim(\mathcal{R}(K_n^*)) = n+1$.

Since $f_\alpha^{\delta,n} \in \mathcal{R}(K_n^*K_n) = \mathcal{R}(K_n^*) = \operatorname{span}\{K^*\hat{L}_j, i = 0,1,\dots,n\}$ (see (26)), $f_\alpha^{\delta,n}$ can be expanded as

$$f_\alpha^{\delta,n} = \sum_{j=0}^{n} a_j K^*\hat{L}_j. \tag{50}$$

Then, equation (47) takes the form

$$\sum_{j=0}^{n}\left(\alpha I + a_j \sum_{i=0}^{n}\langle KK^*\hat{L}_i, \hat{L}_j\rangle_{L^2(a,b)}\right)K^*\hat{L}_j = \sum_{j=0}^{n}\langle g^\delta, \hat{L}_j\rangle_{L^2(a,b)}K^*\hat{L}_j. \tag{51}$$

For notational convenience and simplicity, we denote

$$\overrightarrow{a} = (a_0,\dots,a_n)^\perp \in \mathbb{R}^{n+1}, \tag{52}$$

$$\overrightarrow{g_n^\delta} = \left(\int_a^b g^\delta(t)\hat{L}_0(t)\,dt,\dots,\int_a^b g^\delta(t)\hat{L}_n(t)\,dt\right)^\perp \in \mathbb{R}^{n+1}, \tag{53}$$

$$b_{ij} = \langle KK^*\hat{L}_i, \hat{L}_j\rangle_{L^2(a,b)} = \int_a^b\int_a^b\int_a^b k(s,t)k(t,\tau)\hat{L}_i(s)\hat{L}_i(t)\,ds\,dt\,d\tau, \quad i,j = 0,\dots,n, \tag{54}$$

$$\mathbf{B} = (b_{ij}) \in \mathcal{M}_{n+1}(\mathbb{R}), \qquad \mathbf{A}_n(\alpha) = \alpha I_{n+1} + \mathbf{B}. \tag{55}$$

Now, to determine the unknown coefficients $(a_j)_{j=0}^n$, we must solve the linear algebraic system

$$\mathbf{A}_n(\alpha)\overrightarrow{a} = \overrightarrow{g_n^\delta}. \tag{56}$$

Proposition 4.1 *The linear system* (56) *has a unique solution* $\overrightarrow{a_\alpha^{n,\delta}}$ *for every* $\overrightarrow{g_n^\delta} \in \mathbb{R}^{n+1}$.

Proof Let

$$S(f,g) = \langle K^*Kf, g\rangle_{L^2(a,b)}, \quad f,g \in L^2(a,b).$$

We have

$$S(f,g) = \langle K^*Kf, g\rangle_{L^2(a,b)} = \langle f, K^*Kg\rangle_{L^2(a,b)} = \langle K^*Kg, f\rangle_{L^2(a,b)} = S(g,f)$$

and

$$S(f,g) = \langle K^*f, K^*g\rangle_{L^2(a,b)} \geq 0,$$

that is, $S(\cdot,\cdot)$ is a positive symmetric bilinear form. Hence, $\mathbf{B} = (b_{ij})_{0\leq i,j\leq n} = (S(\hat{L}_i,\hat{L}_j))_{0\leq i,j\leq n}$ is a positive symmetric matrix, and for any $\alpha > 0$, the matrix $\mathbf{A}_n(\alpha)$ of system (56) is invertible. Therefore, this system is uniquely solvable. □

The aim of this part is to derive the convergence and error bound for $\|f - f_\alpha^{n,\delta}\|_{L^2(a,b)}$. To do this, we split the error into three parts:

$$\bigl(f_\alpha^{n,\delta} - f\bigr) = \bigl(f_\alpha^{n,\delta} - f_\alpha^n\bigr) + \bigl(f_\alpha^\delta - f\bigr) + \bigl(f_\alpha^n - f_\alpha^\delta\bigr).$$

Using (37), (41), and the triangle inequality, we can write

$$\Delta_1 = \bigl\|f_\alpha^{n,\delta} - f^n\bigr\|_{L(a,b)} = \bigl\|\bigl(\alpha I + K_n^*K_n\bigr)^{-1}K_n^*\bigl(g_n^\delta - g_n\bigr)\bigr\|_{L(a,b)} \leq \frac{\delta}{2\sqrt{\alpha}}, \tag{57}$$

$$\begin{aligned}\Delta_2 &= \bigl\|f_\alpha^\delta - f\bigr\|_{L(a,b)} \leq \bigl\|f_\alpha^\delta - f_\alpha\bigr\|_{L(a,b)} + \|f_\alpha - f\|_{L(a,b)} \\ &\leq \bigl\|\bigl(\alpha I + K^*K\bigr)^{-1}K^*\bigl(g^\delta - g\bigr)\bigr\|_{L(a,b)} + \|f_\alpha - f\|_{L(a,b)} \\ &\leq \frac{\delta}{2\sqrt{\alpha}} + \|f - f_\alpha\|_{L(a,b)},\end{aligned} \tag{58}$$

$$\begin{aligned}\Delta_3 &= \bigl\|f_\alpha^n - f_\alpha^\delta\bigr\|_{L(a,b)} \leq \bigl\|f_\alpha^n - f_\alpha\bigr\|_{L(a,b)} + \bigl\|f_\alpha - f_\alpha^\delta\bigr\|_{L(a,b)} \\ &\leq \frac{\delta}{2\sqrt{\alpha}} + \bigl\|f_\alpha^n - f_\alpha\bigr\|_{L(a,b)}.\end{aligned} \tag{59}$$

Now, by (36) the quantity $\|f_\alpha^n - f_\alpha\|_{L(a,b)}$ can be estimated as follows:

$$\begin{aligned}\bigl\|f_\alpha^n - f_\alpha\bigr\|_{L(a,b)} &= \bigl\|\bigl(\alpha I + K_n^*K_n\bigr)^{-1}K_n^*g_n - \bigl(\alpha I + K^*K\bigr)^{-1}K^*g\bigr\|_{L(a,b)} \\ &= \bigl\|\bigl(\alpha I + K_n^*K_n\bigr)^{-1}K_n^*K_nf - \bigl(\alpha I + K^*K\bigr)^{-1}K^*Kf\bigr\|_{L(a,b)} \\ &= \bigl\|\mathbb{R}_n(\alpha)h - \mathbb{R}(\alpha)f\bigr\|_{L^2(a,b)} \\ &\leq \bigl\|(A - A_n)(\alpha + A)^{-1}f\bigr\|_{L^2(a,b)}.\end{aligned} \tag{60}$$

Combining (57), (58), (59), and (60), we derive

$$\bigl\|f_\alpha^{n,\delta} - f\bigr\|_{L(a,b)} \leq \frac{3\delta}{2\sqrt{\alpha}} + \bigl\|(A - A_n)(\alpha + A)^{-1}f\bigr\|_{L^2(a,b)} + \|f - f_\alpha\|_{L(a,b)}. \tag{61}$$

Consequently, we have the following theorem.

Theorem 4.1 *Let us assume that* $f = Au \in \mathcal{R}(A)$. *Then, under assumptions* (H1), (H2), *and* (H3), *we have the estimate*

$$\bigl\|f_\alpha^{n,\delta} - f\bigr\|_{L(a,b)} \leq \frac{3\delta}{2\sqrt{\alpha}} + \bigl(\varepsilon(n) + \alpha\bigr)\|u\|_{L^2(a,b)}, \tag{62}$$

where $\varepsilon(n) = \frac{c}{n^r}\|K\|(b-a)M_{r,0}$.

4.1 An a posteriori parameter choice strategy

In this section, we consider the determination of $\alpha(\delta)$ from the discrepancy principle of Morozov. The discrepancy principle (DP) suggests computing $\alpha(\delta) > 0$ such that

$$\|Kf_\alpha^{n,\delta} - g^\delta\|_{L^2(a,b)} = \delta. \tag{63}$$

In this work, we consider a more general class of the damped Morozov principle given by

$$\|Kf_\alpha^{n,\delta} - g^\delta\|_{L^2(a,b)} + \alpha^\eta \|f_\alpha^{n,\delta}\|_{L^2(a,b)}^2 = \delta^2, \tag{64}$$

where $\eta \in [1,\infty]$. Obviously, the classical Morozov principle (63) is a particular case of the damped case with $\eta = \infty$.

In [40, 41], the authors propose a cubically convergent algorithm for choosing a reasonable regularization parameter. This algorithm is summarized as follows.

Algorithm of the cubic Morozov discrepancy principle (CMDP)

Step 1. Input $\alpha_0 > 0$, $\delta > 0$, ϵ(tolerance) > 0, $l_{\max}$, set $l := 0$.
Step 2. Compute $f_{\alpha_l}^{n,\delta}$, $\frac{d}{d\alpha} f_{\alpha_l}^{n,\delta}$, and $\frac{d^2}{d\alpha^2} f_{\alpha_l}^{n,\delta}$.
Step 3. Compute $\Phi(\alpha_l)$, $\Phi'(\alpha_l)$, and $\Phi''(\alpha_l)$ from formula (65).
Step 4. Solve for α_{l+1} from iterative formulas (65), (70), and (71).
Step 5. If $|\alpha_{l+1} - \alpha_l| \le \epsilon$ or $l = l_{\max}$, STOP; otherwise, set $l = l + 1$, GOTO step 2.

$$\Phi(\alpha) = \|Kf_\alpha^{n,\delta} - g^\delta\|_{L^2(a,b)} + \alpha^\eta \|f_\alpha^{n,\delta}\|_{L^2(a,b)}^2 - \delta^2 \tag{65}$$

and

$$\alpha_{l+1} = \alpha_l - \frac{2\Phi(\alpha_l)}{\Phi'(\alpha_l) + (\Phi'(\alpha_l)^2 - 2\Phi(\alpha_l)\Phi''(\alpha_l))^{\frac{1}{2}}}. \tag{66}$$

Now, we present an alternate way to calculate $\Phi'(\alpha)$ and $\Phi''(\alpha)$ in algorithm (CMDP). Let $G(\alpha)$ denote the function

$$G(\alpha) = \|Kf_\alpha^{n,\delta} - g^\delta\|_{L^2(a,b)} + \alpha \|f_\alpha^{n,\delta}\|_{L^2(a,b)}^2 = \psi(\alpha) + \alpha\phi(\alpha), \tag{67}$$

where

$$\psi(\alpha) = \|Kf_\alpha^{n,\delta} - g^\delta\|_{L^2(a,b)}, \qquad \phi(\alpha) = \|f_\alpha^{n,\delta}\|_{L^2(a,b)}^2.$$

The first derivative of $G(\alpha)$ (see [42]) is given by

$$G'(\alpha) = \phi(\alpha). \tag{68}$$

Using (67) and (68), we get

$$G'(\alpha) = \phi(\alpha) = \psi'(\alpha) + \phi(\alpha) + \alpha\phi'(\alpha),$$

which implies that

$$\psi'(\alpha) = -\alpha\phi'(\alpha) \tag{69}$$

and

$$\begin{aligned}\Phi'(\alpha) &= \frac{d}{d\alpha}\bigl(\psi(\alpha) + \alpha^\eta\phi(\alpha) - \delta^2\bigr)\\ &= \psi'(\alpha) + \eta\alpha^{\eta-1}\phi(\alpha) + \alpha^\eta\phi'(\alpha)\\ &= -\alpha\phi'(\alpha) + \eta\alpha^{\eta-1}\phi(\alpha) + \alpha^\eta\phi'(\alpha).\end{aligned}$$

Thus, it follows that

$$\Phi'(\alpha) = \bigl(\alpha^\eta - \alpha\bigr)\phi'(\alpha) + \eta\alpha^{\eta-1}\phi(\alpha) \tag{70}$$

and

$$\Phi''(\alpha) = \bigl(\alpha^\eta - \alpha\bigr)\phi'(\alpha) + \bigl(2\eta\alpha^{\eta-1} - 1\bigr)\phi'(\alpha) + \eta(\eta-1)\alpha^{\eta-2}\phi(\alpha), \tag{71}$$

where

$$\phi'(\alpha) = 2\biggl\langle\frac{d}{d\alpha}f_\alpha^{n,\delta}, f_\alpha^{n,\delta}\biggr\rangle_{L^2(a,b)}$$

and

$$\phi''(\alpha) = 2\biggl(\biggl\langle\frac{d^2}{d\alpha^2}f_\alpha^{n,\delta}, f_\alpha^{n,\delta}\biggr\rangle_{L^2(a,b)} + \biggl\|\frac{d}{d\alpha}f_\alpha^{n,\delta}\biggr\|_{L^2(a,b)}^2\biggr).$$

In our case, using (50), we can write

$$\frac{d^m}{d\alpha^m}f_\alpha^{n,\delta} = \sum_{j=0}^{n}\frac{d^m}{d\alpha^m}a_j(\alpha)K^*\hat{L}_j = \biggl\langle\frac{d^m}{d\alpha^m}\overrightarrow{a(\alpha)}, \overrightarrow{Y}\biggr\rangle, \quad m \geq 1, \tag{72}$$

where

$$\begin{aligned}\frac{d^m}{d\alpha^m}\overrightarrow{a(\alpha)} &= \biggl(\frac{d^m}{d\alpha^m}a_0(\alpha), \frac{d^m}{d\alpha^m}a_1(\alpha), \ldots, \frac{d^m}{d\alpha^m}a_n(\alpha)\biggr)^\perp,\\ \overrightarrow{Y} &= \bigl(K^*\hat{L}_0, K^*\hat{L}_1, \ldots, K^*\hat{L}_n\bigr)^\perp.\end{aligned} \tag{73}$$

It is easy to check that

$$\mathbf{A}_n(\alpha)\frac{d^m}{d\alpha^m}\overrightarrow{a(\alpha)} = -m\frac{d^{m-1}}{d\alpha^{m-1}}\overrightarrow{a(\alpha)}, \quad m \geq 1, \tag{74}$$

where the matrix $\mathbf{A}_n(\alpha)$ is given by (55).

Remark 4.2 We note that formula (74) provides us a practical method to calculate expression (72).

5 Numerical tests

The purpose of this final section is to illustrate this theoretical study with two numerical examples. The numerical experiments are completed with MATLAB.

Example 1

$$\int_0^1 \exp(s^2 t) f(t)\,dt = g(s) = \bigl(\exp(s^2+1)-1\bigr)/(s^2+1)$$

with the exact solution

$$f(t) = \exp(t).$$

Example 2

$$\int_0^{\pi/2} \cos(s^2+3t+1) f(t)\,dt = g(s) = (1/6)\cos(s^2+2) - \pi/4 \sin(s^2)$$

with the exact solution

$$f(t) = \sin(3t+1).$$

Let $\{t_i = a + \frac{(i-1)(b-a)}{N}, i = 1,2,\ldots,N+1\} \subset [a,b]$ the collocation points of the trapezoidal quadrature formula. The trapezoidal quadrature rule associated with these collocation points has the weights $\omega_1 = \omega_{N+1} = \frac{b-a}{2N}$, $\omega_i = \frac{b-a}{N}$, $i = 2,3,\ldots,N$.

We denote by

$$\mathbf{g} = \bigl(g(t_1),\ldots g(t_{N+1})\bigr)^{\perp}$$

the discrete datum of g. Adding a random distributed perturbation (obtained by the Matlab command `randn`) to each data function, we obtain the vector $\mathbf{g}^\delta$:

$$\mathbf{g}^\delta = \mathbf{g} + \varepsilon\,\mathrm{randn}\bigl(\mathrm{size}(\mathbf{g})\bigr),$$

where ε indicates the noise level of the measurement data, and the function "randn(·)" generates arrays of normally distributed random numbers with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "randn(size(g))" returns an array of random entries of the same size as g. The bound on the measurement error δ can be measured in the sense of root mean square error (RMSE) according to

$$\delta = \bigl\|\mathbf{g}^\delta - \mathbf{g}\bigr\|_* = \left(\frac{1}{N+1}\sum_{i=1}^{N+1}\bigl(g(t_i) - g^\delta(t_i)\bigr)^2\right)^{1/2}.$$

The discrete errors $\|e_n\|_\infty$ and $\|e_n\|_2$ are defined by

$$\|e_n\|_\infty = \max_{1\le i\le N+1}\bigl|f(t_i) - f_\alpha^{n,\delta}(t_i)\bigr| \quad \text{and} \quad \|e_n\|_2 = \left[\sum_{i=1}^{N+1} w_i\bigl(f(t_i) - f_\alpha^{n,\delta}(t_i)\bigr)^2\right]^{\frac{1}{2}}.$$

Using the trapezoid rule, we compute

$$(\mathbf{g}_n^\delta)_j = \langle g^\delta, \hat{L}_j\rangle \approx \sum_{i=1}^{N+1} w_i \sum_{k=1}^{N+1} w_k k(s_k, t_i)\mathbf{g}^\delta(s_k)\hat{L}_j(t_i), \qquad j = 0,1,\dots,n,$$
$$\mathbf{g}_n^\delta = \bigl((g_n^\delta)_0, (g_n^\delta)_1, \dots, (g_n^\delta)_n\bigr)^\perp,$$

and

$$b_{ij} = \int_a^b \int_a^b \int_a^b k(s,t)k(t,\tau)\hat{L}_i(s)\hat{L}_i(t)\,ds\,dt\,d\tau$$
$$\approx \mathbf{b}_{ij} = \sum_{m=1}^{N+1} \omega_m \left(\sum_{r=1}^{N+1} w_r \left(\sum_{l=1}^{N+1} \omega_l k(s_l, t_r)k(t_r, \tau_m)\hat{L}_i(s_l)\hat{L}_j(t_r)\right)\right), \quad i,j = 0,\dots,n.$$

Under this notation, we obtain a discrete version of system (56) in the form

$$\mathbf{A}_n(\alpha)\mathbf{a} = \mathbf{g}_n^\delta,$$

and the approximate solution will be calculated by the formula

$$\mathbf{f}_\alpha^{n,\delta}(t_i) \approx \sum_{j=0}^{n} a_j(\alpha)\left(\sum_{r=1}^{N+1} \omega_r k(s_r, t_i)\hat{L}_j(s_r)\right), \quad i = 1,2,\dots,N+1,$$

where

$$\mathbf{B} = (\mathbf{b}_{ij}), \qquad \mathbf{A}_n(\alpha) = (\alpha\mathbf{I}_n + \mathbf{B}), \qquad \mathbf{a} = \bigl(a_0(\alpha), a_1(\alpha), \dots, a_n(\alpha)\bigr)^\perp.$$

Appendix: Tables: Example 1 and Example 2 (discrete data)

Figures 1-8 show the comparison between the exact solution and its computed approximation for different values of n ($n = 2, 3$).

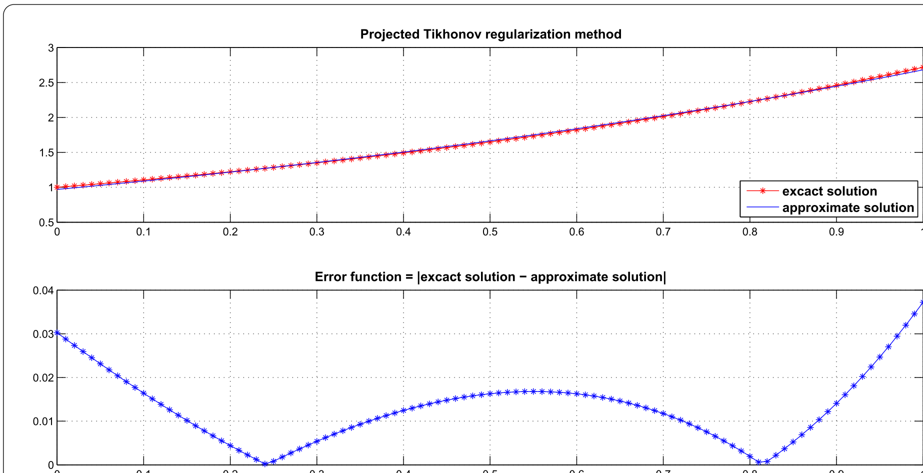

Figure 1 Example 1, noise level = 1/1,000, $n = 2$.

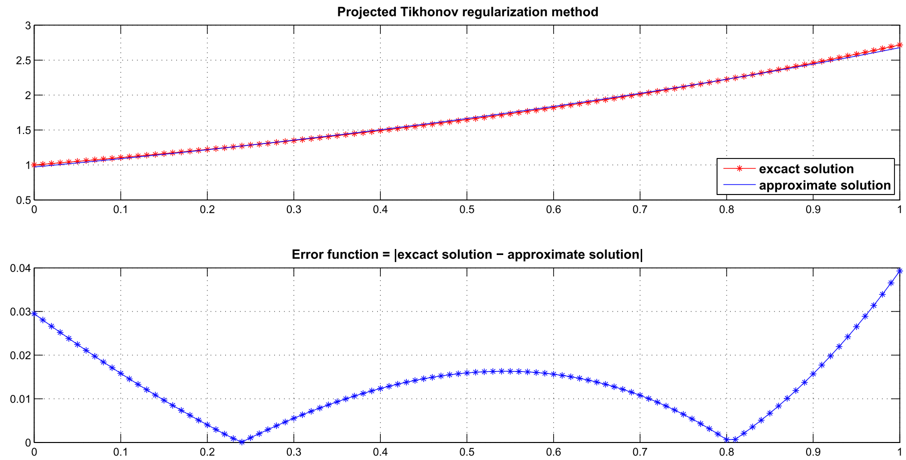

Figure 2 Example 1, noise level = 1/100, $n = 2$.

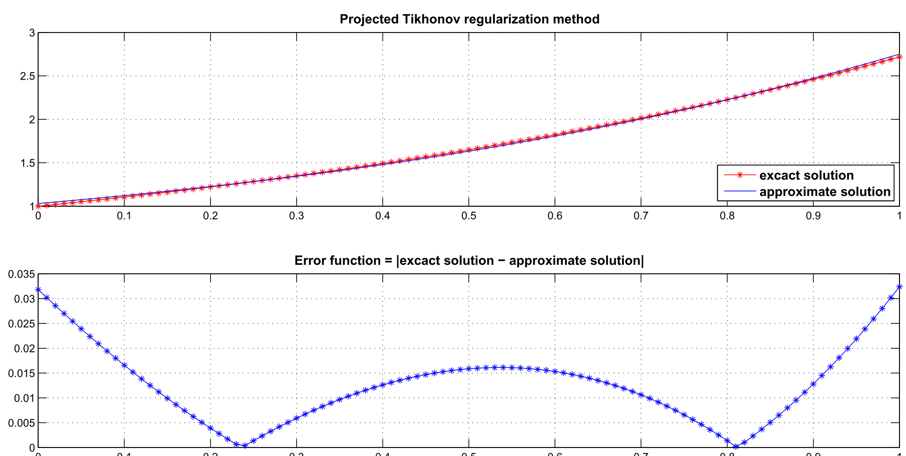

Figure 3 Example 1, noise level = 1/1,000, $n = 3$.

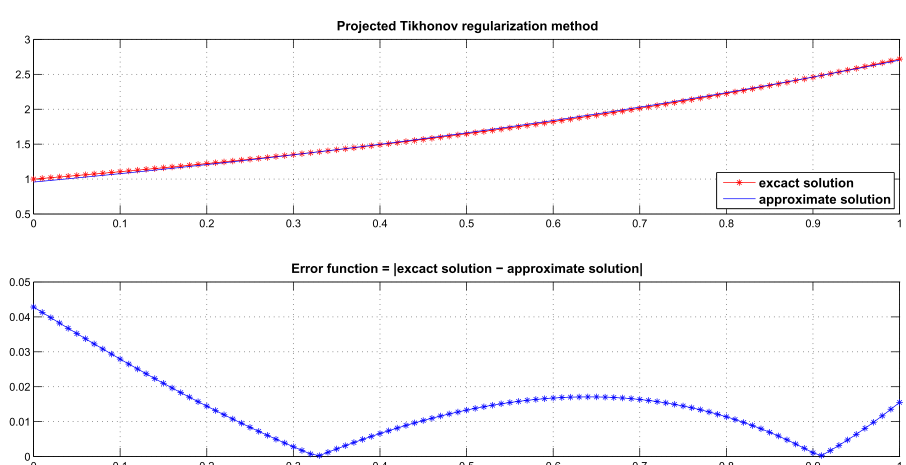

Figure 4 Example 1, noise level = 1/100, $n = 3$.

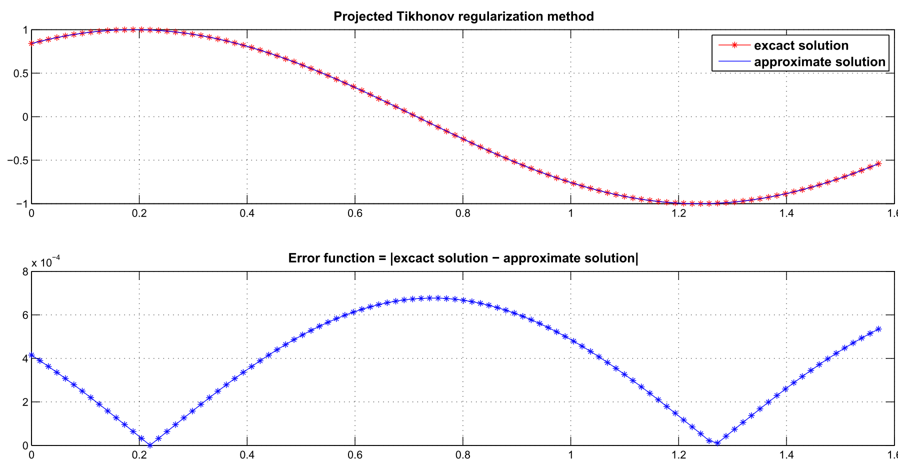

Figure 5 Example 2, noise level = 1/1,000, $n = 2$.

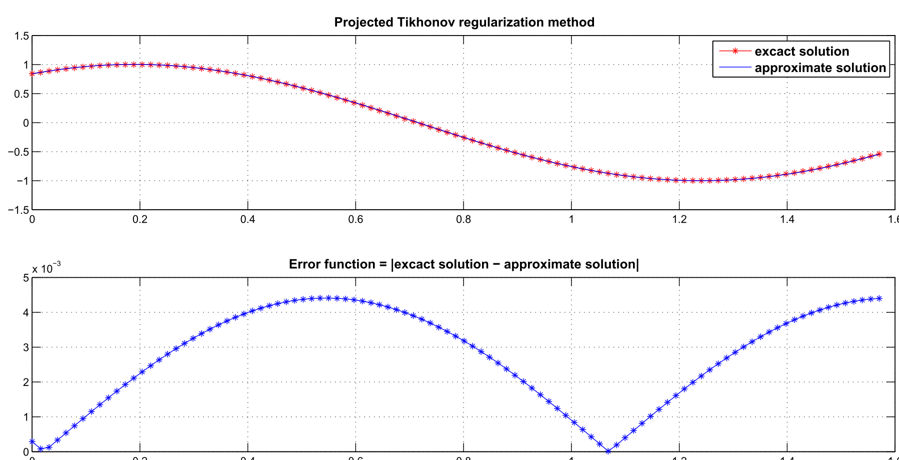

Figure 6 Example 2, noise level = 1/100, $n = 2$.

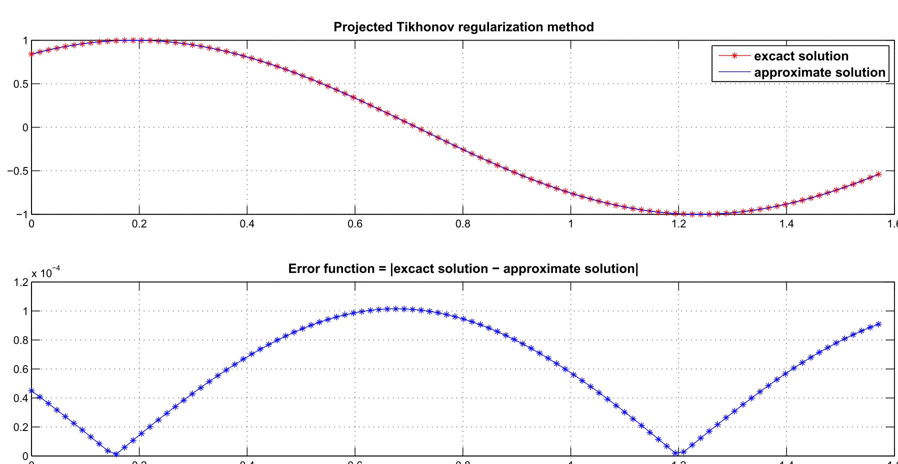

Figure 7 Example 2, noise level = 1/1,000, $n = 3$.

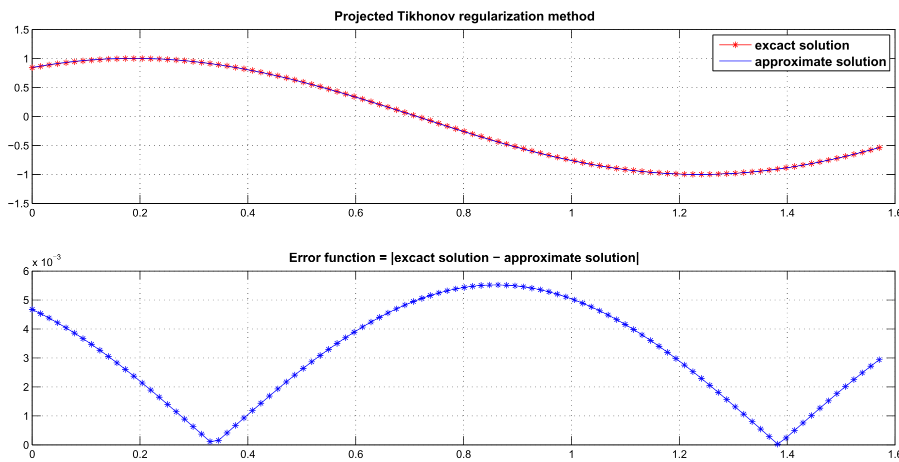

Figure 8 Example 2, noise level = 1/100, $n = 3$.

Table 1 Example 1, $n = 2$

ε	α	ℓ^2-norm	ℓ^∞-norm	re (relative error)
0.001	1.0584519676359 53e–010	0.015010783744725	0.035613080391689	8.343729340481704e–004
0.01	1.523132088372481e–006	0.015011143017205	0.035448296130840	8.343929041735275e–004

Table 2 Example 1, $n = 3$

ε	α	ℓ^2-norm	ℓ^∞-norm	re (relative error)
0.001	3.386965172845820e–009	0.003471711713518	0.007531359104993	1.929747532066797e–004
0.01	4.363395697816134e–007	0.011051454981111	0.027181028421710	6.142940352019532e–004

Table 3 Example 2, $n = 2$

ε	α	ℓ^2-norm	ℓ^∞-norm	re (relative error)
0.001	4.848918710302366e–008	3.516777971379156e–004	3.637947670728225e–004	4.534822996062119e–005
0.01	6.575353855416842e–006	0.004454265966245	0.005633164279727	5.743697184950506e–004

Table 4 Example 2, $n = 3$

ε	α	ℓ^2-norm	ℓ^∞-norm	re (relative error)
0.001	5.187878275981052e–007	2.337236516213294e–004	2.682894481292331e–004	3.013825151095230e–005
0.01	1.901137499862827e–005	0.002424879642149	0.002729060667431	3.126839411925939e–004

Conclusion. From Tables 1-4 we see that the numerical results agree with the theoretical results.

The projected Tikhonov regularization method developed and used in this investigation to solve the Fredholm integral equations of the first kind is very simple and effective, owing to the fact that the dimension of the subspace of projection is very small ($n = 2, 3$); moreover, the regularized solution remains stable for a strong noise ($\varepsilon = 1/100$) and for regular data.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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